RIEMANN FUNCTIONS FOR A SYSTEM OF HYPERBOLIC FORM IN THREE INDEPENDENT VARIABLES

BY

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ABSTRACT

Functions are defined which permit the solution of a special hyperbolic system to be expressed as a quadrature of its initial data over the initial surface.

1. **Introduction.** In this paper Riemann functions (R.F.) are defined for systems of partial differential equations of the type

$$L(U) = (D - A)\,U = 0 \tag{1.1}$$

where

$$U \equiv (U^1, \cdots, U^N), \quad A \equiv (a_{ij}(x)), \quad x \equiv (x_1, x_2, x_3)$$

$$D \equiv (D_1, \cdots, D_N), \quad D_i \equiv \sum_{j=1}^{j=3} \alpha_{ij} \frac{\partial}{\partial x_j}$$

and the direction numbers $\vec{\alpha}_i \equiv (\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3})$ are constant, distinct and oblique to the initial data surface.

We assume that initial data is specified on an initial data surface, θ, and for simplicity and without loss of generality chose for θ the hyperplane $x_1 = 0$. We shall show that the value of U at any point P, not on $x_1 = 0$ is a quadrature of its initial data and the R.F. over a subset of the initial hyperplane.

Also for purposes of simplicity and visualization the further non-restrictive hypothesis are made that P is in the upper half plane, and that the vectors $\vec{\alpha}_i$, $i = 1, \cdots, N$ are direction cosines and have positive projections in the x_1 direction i.e. the vectors $\vec{\alpha}_i$ point in the general direction of the positive x_1 axis. A restrictive assumption we make is that no three vector $\vec{\alpha}_i$ through P are coplanar; this assumption will finally be removed.

Riemann functions were defined for systems similar to (1.1) in [1, 2, 3] and the

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techniques used here are a synthesis of ideas introduced in those papers. Although this paper is almost completely selfcontained, a familiarity with [1, 2, 3] should be helpful.

2. **Orientation and notation.** The R.F. for each component of U is a set of vector valued functions. Each member of the set is a solution to the adjoint operator to (1.1), L^*, defined in a domain D^i. These domains are three dimensional conical subregions in the interior of the backward facing ray cone that is formed by taking the convex hull of the backward (negative x_1 direction) characteristics (the characteristic C_l is the line in the direction of $\vec{\alpha}_l$) issuing from P. In order to describe more exactly these conical subregions of the backward ray cone it is convenient to introduce the concept of a wedge. A wedge is simply the planar area between two backward characteristics issuing from P. The wedge formed by the backward characteristics C_p and C_q issuing from P is denoted ω_{pq}. Sometimes ω_{pq} is used to denote only that part of ω_{pq} between P and the initial hyperplane; the exact meaning of ω_{pq} being clear from the context. The wedges generated by every pair of backward characteristics issuing from P form the sides of the backward ray cone and divide it into the subcones D^i. Two points lie in the same subcone if the line segment connecting them does not intersect a wedge. Sometimes D^i is also used to denote only that part of the subcone between P and $x_1 = 0$.

For an arbitrary component U^K of U we will define in each subcone D^i a solution, W^i, of $L^* = 0$ and together these solutions will comprise the set of R.F. for the component U^K of U. Throughout the paper capital K denotes the index of the arbitrary component of U for which R.F. are being defined.

Cauchy data for the R.F. is defined on the wedges that form the boundaries of the domains D^i. The motivation for the specification of this Cauchy data is explained in the next section.

3. **Specification of the Cauchy data.** The value of U^K at P can be expressed in terms of quadratures of U and auxillary functions over sections of the wedges and the initial hyperplane. Thus, by employing Green's identity

$$\int_{C_K} V(D_K U^K - a_{KK} U^K)\, ds + \int_{C_K} U^K (D_K V + a_{KK} V) ds = V U^K \Big|_{P'_K}^{P}, \tag{3.1}$$

In equation (3.1), P'_K is the point where the characteristic C_K intersects the initial data plane and V is an, as yet, unspecified function. When the Kth equation of (1.1) is substituted into (3.1) and V is chosen to be the solution of

$$D_K V + a_{KK} V = 0 \tag{3.2}$$

and

$$V_{(P)} = 1$$

then we have

$$\int_{C_K} V\Big(\sum_{l\neq K} a_{Kl}U^l \Big)\, ds = VU^K \Big|_{P'_K}^{P} \tag{3.3}$$

Green's identity is also applied to the area ω_{Kj}, $j \neq K$. Using the coordinate system formed by taking C_K and C_j as coordinate axis we have

$$\begin{aligned}(3.4)\quad &\int_{\omega_{Kj}} \{Z^K(D_K U^K - a_{KK}U^K - a_{Kj}U^j) + Z^j(D_j U^j - a_{jK}U^K - a_{jj}U^j)\}d\omega \, + \\ &+ \int_{\omega_{Kj}} \{U^K(D_K Z^K + a_{KK}Z^K + a_{jK}Z^j) + U^j(D_j Z^j + a_{Kj}Z^K + a_{jj}Z^j)\}d\omega \\ &\qquad = C_{Kj}\Big\{\oint Z^K U^K ds_j + \oint Z^j U^j ds_K\Big\}\end{aligned}$$

where the line integrals are taken around the boundary of ω_{Kj}; $d\omega = C_{Kj}ds_j ds_K$ is an element of area of ω_{Kj}; ds_j and ds_K are elements of arc length along C_K and C_j; and C_{Kj} is the sine of the angle between C_K and C_j in the wedge ω_{Kj} at P. Furthermore,

$$\oint Z^K U^K ds_j = \int_{C_j} Z^K U^K ds_j - \int_{\overline{P'_K P'_j}} Z^K U^K ds_j \tag{3.5}$$

the first line integrand on the right hand side being evaluated over C_j and the second integral over $P'_K P'_j$, which is the line segment between P'_K and P'_j lying in the initial plane. The signs on the right hand side of (3.5) depend on the orientation of C_K with C_j. Similarly

$$\oint Z^j U^j ds_K = \int_{C_K} Z^j U^j ds_K - \int_{\overline{P'_K P'_j}} Z^j U^j ds_K \tag{3.6}$$

Chosing Z^K and Z^j to be solutions, in ω_{Kj}, of

$$\begin{aligned} D_K Z^K + a_{KK}Z^K + a_{jK}Z^j &= 0 \\ D_j Z^j + a_{Kj}Z^K + a_{jj}Z^j &= 0 \end{aligned} \tag{3.7}$$

satisfying

$$\begin{aligned} Z^K &= 0 \quad \text{on } C_j \\ Z^j &= V a_{Kj} C_{Kj}^{-1} \quad \text{on } C_K \end{aligned} \tag{3.8}$$

and then combining these equations with (3.4) gives

$$(3.9)\quad \iint_{\omega_{Kj}} \Big\{Z^K[D_K U^K - a_{KK}U^K - a_{Kj}U^j] + Z^j[D_j U^j - a_{jK}U^K - a_{jj}U^j]\Big\}\,d\omega$$
$$= \int_{C_K} V a_{Kj} U^j ds_K \;+\; \left\{\begin{array}{l}\text{line integrals of } Z'^s \text{ and } U'^s\\ \text{evaluated along } \overline{P'_K P'_j}\end{array}\right\}$$

When the Kth and jth equations of (1.1) are inserted in the integrands of the left hand side of (3.9) that equation becomes

$$(3.10)\quad \iint_{\omega_{Kj}} \sum_{l\neq K,j} (Z^K a_{Kl} + Z^j a_{jl}) U^l d\omega_{Kj} = \int_{C_K} V a_{Kj} U^j ds_K$$
$$+ \left\{\begin{array}{l}\text{line integrals of } Z'^s \text{ and } U'^s\\ \text{evaluated along } \overline{P'_K P'_j}\end{array}\right\}$$

After equation (3.10) is summed for all j but $j = K$ and this sum substituted into (3.3) the result is

$$(3.11)\quad \sum_{j\neq K} \iint_{\omega_{Kj}} \sum_{l\neq K,j} (Z^K a_{Kl} + Z^j a_{jl}) U^l d\omega_{Kj} = U^K_{(P)} - V_{(P'_K)}\, U^K_{(P'_K)}$$
$$+ \sum_{j\neq K} \left\{\begin{array}{l}\text{line integrals of } Z'^s \text{ and } U'^s\\ \text{evaluated over } \overline{P'_K P'_j}\end{array}\right\}$$

This formula plays a key role in the definition of R.F. and its significance will be seen after Green's identity has been written out for each region D^j and these identities collected. Thus,

$$(3.12)\quad \sum_j \iiint_{D_j} W^j L(U) + U L^*(W^j)]\, d_{x_1} \cdots d_{x_3} = \sum_j \iint_{S_j} W^j * U\, dA$$

In (3.12) W^j are, as yet, unspecified vector valued functions defined in the domain D^j; L^* is the adjoint operator of L; S_j is the boundary D^j;

$$W^j * U = W_1^j U^1 \vec{\alpha}_1 \cdot \vec{N} + \cdots W_N^j U^N \vec{\alpha}_N \cdot \vec{N}$$

and $\vec{N}$ is the outwardly directed normal on S_j. The surface integrals in (3.12) can be further decomposed into a quadrature over the parts of S_j consisting of wedges and the parts lying in the initial plane

$$(3.13)\quad \sum_j \iint_{S_j} W^j * U dA = \sum_{\omega'^s} \iint_{\omega} W^j * U d\omega + \sum_j \iint_{S_j \cap \theta} W^j * U dA$$

If in D^j

$$(3.14)\quad L^*(W^j) = 0$$

and Cauchy data for the functions W^j is assigned on the ω'^s so that

$$(3.15)\qquad \sum_{\omega'^s} \iint_{\omega} W^j * U d\omega = \sum_{j \neq K} \iint_{\omega_{Kl}} \sum_{l \neq j,K} (Z^K a_{Kl} + Z^j a_{jl}) U^l d\omega$$

then it is seen by combining (3.11)–(3.15) that

$$(3.16)\qquad U^K_{(P)} = V_{(P'_K)} U^K_{(P'_K)} + \sum_{j \neq K} \left\{ \begin{array}{l} \text{line integrals of } Z'^s \text{ and } U'^s \\ \text{evaluated over } \overline{P'_K P'_j} \end{array} \right\} + \sum_j \left\{ \begin{array}{l} \text{surface integrals} \\ \text{of } W^{j's} \text{ and } U'^s \\ \text{evaluated over } S_j \cap \theta \end{array} \right\}$$

Hence the value of $U^K_{(P)}$ is expressed in terms of its Cauchy data on θ; and the functions V, Z'^s and $W^{j's}$ are R.F. Motivated by these considerations, our objective is to assign the Cauchy data for the functions W^j so that (3.15) holds.

The terms on the left hand side of (3.15) can be separated into two classes, depending on whether the surface integration is over a wedge from the set of wedges, $\{A\}$, that form the sides of the backward ray cone or whether the integration is over a wedge from the set $\{B\}$ of wedges which lie in the interior of the backward ray cone. The terms in the latter set can be paired naturally. Two terms form a pair if they are integrations over opposite sides of the same wedge. These terms arise from applying Green's identity to the two regions D^p and D^q that lie on opposite sides of a wedge ω_{pq} from the set $\{B\}$.
Thus,

$$(3.17)\qquad \sum_{\omega'^s} \iint_{\omega} W^j * U d\omega = \sum_{\omega \in \{A\}} \iint_{\omega} W^j * U d\omega$$
$$\sum_{\omega_{-} \in \{B\}} \iint_{\omega_{-}} \sum_{l} (W^p_l \vec{\alpha}_l \cdot \vec{N}^p + W^q_l \vec{\alpha}_l \cdot \vec{N}^q) U^l d\omega_{pq}$$

where $\vec{N}^p$ and $\vec{N}^q$ are the outward normals from the regions D^p and D^q respectively on ω_{pq}. Since we have when $\vec{N}$ is a normal to ω_{pq} that

$$(3.18)\qquad \begin{array}{ll} \vec{\alpha}_l \cdot \vec{N} = 0, & l = p,q \\ \vec{\alpha}_l \cdot \vec{N} \neq 0, & l \neq p,q \end{array}$$

Equation (3.17) can be reduced to

$$(3.19)\qquad \sum_{\omega'^s} \iint_{\omega} W^j * U d\omega = \sum_{\omega_{-} \in \{A\}} \left[\iint_{\omega_{-}} \sum_{l \neq p,q} W^j_l (\vec{\alpha}_l \cdot \vec{N}) U^l d\omega_{pq} \right]$$
$$+ \sum_{\omega \in \{B\}} \left[\iint_{\omega_{pq}} \sum_{l \neq p,q} (W^p_l \vec{\alpha}_l \cdot \vec{N}^p + W^q_l \vec{\alpha}_l \cdot \vec{N}^q) U^l d\omega_{pq} \right]$$

The Cauchy data for the set of vectors W^j, which are defined in the regions D^j, will be chosen in a such way that the right hand sides of (3.19) and (3.15) are equal. A simple way of accomplishing this is to specify the vectors W^j on the ω'^s so that the coefficients of the functions U^l in the integrands of those two expressions are equal. A comparison of these coefficients yields the analytical prescription for the conditions that the W^j must satisfy (for understanding these conditions it is helpful to keep in mind that the subindices on ω_{pq} indicate that the wedge is formed by the characteristics C_p and C_q intersecting at P): If W_p is defined in a subcone D^p one of whose sides is ω_{pq} and $\omega_{pq} \in \{A\}$ and also $\omega_{pq} \in \{C\} \equiv \{\cup_j \omega_{Kj}\}$ then for all points on ω_{pq}, the function W^p must satisfy

$$W_l^p \vec{\alpha}_l \cdot \vec{N} = Z^p a_{pl} + Z^q a_{ql} \tag{3.20}$$

for all $l \neq p, q$; but if $\omega_{pq} \in \{A\}$ and $\omega_{pq} \notin \{C\}$ then

$$W_l^p (\vec{\alpha}_l \cdot \vec{N}) = 0 \tag{3.21}$$

for all $l \neq p, q$. Similarly if W^p and W^q are defined in the subcones D^p and D^q that are separated by ω_{pq} and $\omega_{pq} \in \{B\} \cap \{C\}$ then at all points on ω_{pq} the vectors W^p and W^q must satisfy

$$W_l^p \vec{\alpha}_l \cdot \vec{N}^p + W_l^q \vec{\alpha}_l \cdot \vec{N}^q = Z^p a_{pl} + Z^q a_{ql} \tag{3.22}$$

for all $l \neq p, q$; but if $\omega_{pq} \in \{B\}$ and $\omega_{pq} \notin \{C\}$ then

$$W_l^p \vec{\alpha}_l \cdot \vec{N}^p + W_l^q \vec{\alpha}_l \cdot \vec{N}^q = 0 \tag{3.23}$$

for all $l \neq p, q$.

If (3.20–3.23) are satisfied then (3.15) holds; and if in addition the W^j satisfy (3.14) then we get equation (3.16). A system of integral equations for the functions W^j will be constructed such that the solution of the system satisfies (3.20–3.23) and (3.14).

4. **Construction of the system of integral equations.** We will obtain the system of integral equations by illustrating how a typical equation is derived. A single equation will be associated with each unknown, W_l^i so that the number of unknowns and equations in the system are the same. Since the numbering of the subcones D^i is arbitrary there is no loss in generality in taking $i = 1$. To derive the equation associated with W_l^1 at an arbitrary interior point $P_1 \in D^1$ draw through P_1 the characteristic C_l. This characteristic intersects successively wedges $W_1, \cdots, W_{s-1}$ from the set $\{B\}$ terminating at a point P_s on a wedge W_s that forms part of the boundary of the backward ray cone. Let the points of intersection of C_l with these wedges be ordered according to their distance from P_1 and denote them by $P_2, \cdots, P_s$.

Similarly, in passing from P_1 to P_s the characteristic C_l passes successively through the subcones $D^1, \cdots D^{s-1}$. If P_m and P_n (where P_n is in the positive direction from P_m) are any two points inside or on the boundary of a subcone D^r then by integrating the system (3.15) there results

$$W_l^r(P_m) = W_l^r(P_n) + \int_{P_m}^{P_n} \left(\sum_j a_{lj} W_j^r \right) ds \tag{4.1}$$

In the case of interest to us P_m and P_n will belong to $\{P_1, \cdots, P_s\}$ and will hence be on opposite sides of the boundary of a subcone $D^n \in \{D^1, \cdots, D^s\}$. The integral equation for W_l^1 is obtained by combining (3.20–3.23) and (4.1).

Thus putting $P_m = P_1, P_n = P_2$ in (4.1) it becomes

$$W_l^1(P_1) = W_l^1(P_2) + \int_{P_1}^{P_2} \left(\sum_j a_{lj} W_j^1 \right) ds \tag{4.2}$$

Since $P_2 \in \omega_1 \equiv \omega_{p_1, q_1}$, we get by using (3.22) (for the sake of argument it is assumed $\omega_1 \in \{C\}$) in (4.2) that

$$\begin{aligned} W_l^1(P_1) \; = \; & W_l^2(P_2) + (\vec{\alpha}_l \cdot \vec{N}^{q_1})^{-1} (Z^{p_1} a_{p_1 l} + Z^{q_1} a_{q_1 l}) \\ & + \int_{P_1}^{P_2} \left(\sum_j a_{lj} W_j^1 \right) ds \end{aligned} \tag{4.3}$$

where in deducing (4.3) the relation

$$(\vec{\alpha}_l \cdot \vec{N}^{p_1})(\vec{\alpha}_l \cdot \vec{N}^{q_1})^{-1} = -1$$

and (3.18) have also been used. By employing (4.1) with $r = 2$ and $P_m = P_2$, $P_n = P_3$ in the right hand side of (4.3) it becomes

$$\begin{aligned} W_l'(P_1) = W_l^2(P_3) + (\vec{\alpha}_l \cdot \vec{N}^{q_1})^{-1} (Z^{p_1} a_{p_1 l} + Z^{q_1} a_{q_1 l}) \Big|_{P_2} + \int_{P_1}^{P_2} \left(\sum a_{lj} W_j^1 \right) ds \\ + \int_{P_2}^{P3} \left(\sum_j a_{lj} W_j^2 \right) ds \end{aligned} \tag{4.4}$$

Since $P_3 \in \omega_2$ the procedure in deriving (4.4) from (4.2) can be repeated and so on. The final equation for $W_l^1(P_1)$ is

$$\begin{aligned} W_l^1(P_1) = (\vec{\alpha}_l \cdot \vec{N}^{q_1})^{-1} (Z^{P_1} a_{P_1 l} + Z^{q_1} a_{q_1 l}) |_{P_2} + \cdots + \\ + \sum_{m=1}^{s-1} \int_{P_m}^{P_{m+1}} \left(\sum_{j=1}^{N} a_{lj} W_j^m \right) ds \end{aligned} \tag{4.5}$$

The system of equations consisting of equations like (4.5) for all W_l^i, $(i = 1, \cdots)$, $l = 1, \cdots, N$ can be solved by the method of successive approximations in the standard manner. Since the way in which this system of integral equations was derived from (3.20–3.23) and (3.14) is reversible the solution of the system will satisfy those conditions; and as has been demonstrated this implies the functions W^i are R.F. for (1.1).

The preceeding derivation of Equation (4.5) contains a source of ambiguity that was glossed over. The tacit assumption was made that D^l was separated from D^{i+1} along C_l by a unique wedge ω_{i+1} whereas examples are readily constructed where the point at which C_l crosses between D^i and D^{i+1} lies on the line of intersection of two or more wedges. We will show that there is no real ambiguity in this circumstance by extending the prescription for the Cauchy data so as to include these cases. When C_l crosses between two subcones at a point P^* on the line of intersection of two or more wedges then perturb the point P_1 so that C_l crosses between D^i and D^{i+1} at a point P^{**} not on the line of intersection of two wedges. At the perturbed point P^{**} the Cauchy data is unambiguously assigned by the conditions (3.20–3.23). The extension of our prescription is completed by defining the Cauchy data at P^* to be limit of the Cauchy data for P^{**} as P^{**} approaches P^*. This limit is independent of the way P^{**} approaches P^*.

In the previous discussion it was assumed that no three characteristics through P lie in the same plane. This assumption was employed explicitly in (3.18) and implicitly in asserting that the wedges divided the backward ray cone into 3-dimensioned subcones. We will sketch how this restriction can be removed.

A different proceedure must be adopted only when defining the R.F. for a component U^K of U for which C_K is coplanar with more than one other characteristic through P. We consider the representative case of a component U^K of U for which the characteristics $C_1, \cdots, C_K$ through P are coplanar while none of the other characteristics $C_{K+1}, \cdots, C_N$ through P lie in that plane. In order to successfully carry through the method of section 4 for defining the integral equation (4.5) in this case it is necessary to replace equation (3.11) with a more suitable expression for the value of $U^K_{(P)}$. This is because (3.18) is false in the present situation; and hence if the method of section 4 were carried through then equation (4.5) would contain a division by zero.

This difficulty can be circumvented by substituting in place of (3.11) the new expression for $U^K_{(P_1)}$

$$U^K_{(P_1)} = V(P_1')U^K(P_1') + \sum_{j=1}^{K-1} \left\{ \begin{matrix} \text{line integrals of the functions} \\ Z^{il} \text{ and } U \text{ evaluated along } \overline{P_j'P_{j+1}'} \end{matrix} \right\} + \sum_{i=1}^{i=K-1} \left[\iint_{\omega_{i,i+1}} \sum_{j=K+1}^{N} \left\{ \sum_{l=1}^{K} Z^{il} a_{lj} \right\} U^j d\omega \right] \tag{4.6}$$

which is derrived using the methods of [1, 2]. In equation (4.6) the function V is an auxillary function defined in a manner similar to the definition of the given in (3.2) while the functions Z^{il} are constructed in a manner analogous to the construction of the $W^{i's}$ given in section 4.

Equation (4.6) expresses the U^K as a linear functional of the Cauchy data of the functions $U^1, \cdots, U^K$ evaluated along $\overline{P_1'P_K'}$ and surface integrals of $U^{K+1}, \cdots U^N$ evaluated over the wedges $\omega_{i,i+1}$. These latter quadratures can be reexpressed as quadratures of the functions U over θ. The method of proceedure being entirely analogous to that used in going from (3.11) to (4.5) except (4.6) replaces (3.11). The function W^i in this case are defined, of coarse, only in the non-degenerate 3-dimensional subcones. This procedure is feasible when beginning with (4.6) instead of (3.11) because Cauchy data does not need to be assigned for any component W_l^j of W^j on a wedge for which $(\vec{\alpha}_l \cdot \vec{N}) = 0$.

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